An interesting special characteristic of the solidification process under consideration is the fact that the region 3 appears in the form of a sharp edge. This can be seen from the expression $y_* = -\varepsilon^2\delta^* (\beta_2 + \varepsilon\beta_3 + \ldots)$, differentiating which with respect to x we will have $\left.\frac{dy_*}{dx}\right|_{x_0} = -\frac{\nu_g}{U\delta_0^*}\beta_2$, i.e., the angle between a tangent to the surface and the x axis has a finite value.

It must also be added that the formulas obtained (5.5) ($U = U_S$) are valid also in the case where there is a velocity of the external flow. Under these circumstances, the effect of the value and direction of the velocity will appear through the value of $q_0''(0)$. If we set $Pr_g = 1$, from the Crocco integral [7], we can obtain

$$q_0' = \frac{u - U_\infty}{U - U_\infty}\left(1 - \frac{T_\infty}{T_S}\right), \text{ then } q_0''(0) = \mu_g^{-1}\frac{\tau_w}{U - U_\infty}\left(1 - \frac{T_\infty}{T_S}\right).$$

This value can serve for an evaluation of the effect of the velocity of the flow and the temperature of the external medium on the solidification process.

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TRANSONIC FLOWS OF GAS IN LAVAL NOZZLES WITH LOGARITHMIC SPECIAL FEATURES IN THEIR LIMITING CHARACTERISTICS

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In flat Laval nozzles there are three types of asymptotic flows in the neighborhood of the center [1]. This conclusion was reached using the theorem of Brio and Buke [2] with respect to the behavior, near a singular point (image of the limiting characteristic), of the general integral of the ordinary differential equation to which a study of self-similar transonic flows reduced. It has been found that, with some values of the index of the self-similarity, any given integral curve can be analytically prolonged through this singular point, which is a mesh point in the problem of nozzle flows. With the consideration of flows in nozzles with a round transverse cross section, it is useful to consider the same indices which are considered in the theorem. It is well known [3] that, in this case, there is a second possible alternative: None of the integral curves passing through the mesh point, with the exception of an isolated whisker, yield an analytical continuation. This relates also to a whisker of general direction. In other words, a whisker of general direction holds out the possibility of an analytical prolongation with any given self-similarity index, with the exception of those considered in the theorem. In [4], a second asymptotic type of flow in the neighborhood of the center of an axisymmetric nozzle is constructed numerically

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using a whisker of general direction. However, as a result of the calculations, an index $n^* = 2.36532$ (xr^{-n} is an invariant of the self-similar solution) was obtained, which is close to $n_1 = 2.36589$, i.e., to one of those considered in the theorem. We note that the coincidence $n^* = n_1$ signifies the impossibility of the second asymptotic type of flow in the neighborhood of the center of an axisymmetric nozzle. Moreover, with values of n close to n_1, the method used in [4] of starting from a singular point using a power series is incorrect, since all the coefficients, starting from some arbitrary one, tend to infinity with $n \to n_1$. It is obvious that, in axisymmetrical Laval nozzles, only flows with weak discontinuities at the corresponding limiting characteristics can exist, if the self-similarity index n varies in the interval $2 < n < \infty$. The indices n_1, n_2, n_3, to which the theorem reduces in the case of axisymmetrical nozzles, areconsidered below. Along a characteristic arriving at the center of the nozzle, there is propagated a logarithmic discontinuity in the derivatives of the components of the velocity along the coordinates (with $n = n_1$, there is a discontinuity at three, with $n = n_2$ at four, and with $n = n_3$ at five derivatives). After reflection of the singularity from the center of the nozzle, in the case $n = n_1$ there arises a weak discontinuity at the departing limiting characteristic; in the case $n = n_2$, there is a limiting line, not eliminated by a jump in the density; in the case $n = n_3$ there arises a shock wave.

1. Asixymmetric flows of gas in a near-sonic approximation are described by the Cauchy equation

$$-\varphi_x\varphi_{xx} + \varphi_{rr} + (1/r)\varphi_r = 0, \tag{1.1}$$

where x, r are cylindrical coordinates; φ is the potential of the perturbations of a homogeneous sonic flow. For investigation of flows in Laval flows with a round transverse cross section, we consider the Cauchy problem: Find the solution of Eq. (1.1) satisfying the following conditions at the axis of symmetry r = 0:

$$\varphi_x = -A_1|x|^k \ (x < 0);\ \varphi_x = A_2x^k \ (x > 0),\ A_1 > 0,\ A_2 > 0. \tag{1.2}$$

If a shock wave arises in the flow, the solution must satisfy the additional conditions at the wave front

$$\varphi_1 = \varphi_2,\ 2[dx(r)/dr]^2 = \varphi_{x1} + \varphi_{x2},$$

where the subscripts relate to the parameters of the gas on different sides of the front; $x = x(r)$ is the equation of the shock wave.

The Cauchy problem (1.1), (1.2) with $1 < k < 2$ was investigated in [4]. We shall consider below the corresponding solution with

$$\begin{aligned} k = k_1 &= (7 + 2\cdot 21^{1/2})/14 = 1.15465,\\ k = k_2 &= (26 + 25\cdot 2^{1/2})/41 = 1.49647,\\ k = k_3 &= (7 + 91^{1/2})3/28 = 1.77208, \end{aligned} \tag{1.3}$$

where, at the limiting characteristic C_0^-, arriving at the center of the nozzle, there arises a logarithmic singularity in the higher derivatives of the components of the velocity along the coordinates.

2. The Cauchy problem (1.1), (1.2) has the self-similar solution

$$\varphi = r^{3n-2}q(\zeta),\ \zeta = xr^{-n},\ n = 2/(2-k);$$

here the equation of the shock front has the form ζ = const.

The function $q(\zeta)$ satisfies the ordinary differential equation

$$(n^2\zeta^2 - q')q'' - n\zeta(5n-4)q' + (3n-2)^2q = 0. \tag{2.1}$$

To construct the flow in the inlet part of the nozzle between the semiaxis $x < 0$ and the characteristic C_0^-, we must use the integral curve of Eq. (2.1) (we denote it by S), which, in the neighborhood of the point $\zeta = -\infty$, is described by the expansion [3]

$$q = \sum_{i=0}^{\infty} d_i |\zeta|^{(3n-2-2i)/n}, \quad d_0 = nA_1/(3n-2), \tag{2.2}$$

$$d_i = (-2n+1+i)/(4n^3 i^2) \sum_{j=0}^{\infty} (3n-2-2j)(3n-2i+2j)\, d_j d_{i-1-j}.$$

The expansion (2.2) determines the curve S from the point $\zeta = -\infty$ up to some singular point ζ_c, corresponding to a limiting characteristic. The point ζ_c is determined by the equalities

$$n^2\zeta_c^2 - q'(\zeta_c) = 0, \quad n\zeta_c(5n-4)\, q'(\zeta_c) = (3n-2)\, q(\zeta_c)$$

and, with $1 < k < 2$ $(2 < n < \infty)$, is a mesh point.

We write the general integral of Eq. (2.1) in the neighborhood of ζ_c in the form

$$q = \sum_{i=0}^{E} a_i \zeta_c^{3-i} \Delta^i + a_\mu \zeta_c^{3-\mu} \Delta^\mu + \ldots, \quad \Delta = \zeta - \zeta_c, \tag{2.3}$$

$$\mu = (14n - 8)/(7n - 4 - R), \quad R = (25n^2 - 56n + 32)^{1/2},$$

where μ is the power exponent of the first term of the nonregular part; E is the greatest whole number, not exceeding μ; a_μ is an arbitrary constant; the coefficients a_i $(0 \leqslant i \leqslant E)$ have the form $(\mu \neq 4, 5, 6)$

$$a_0 = n^3(5n-4)(3n-2)^{-2}, \quad a_1 = n^2, \quad a_2 = (4 - 3n + R)\, n/4, \tag{2.4}$$

$$a_i = A_i/B_i, \quad A_i = -a_{i-1}[n(i-4)+2]^2 + (i/2)\sum_{j=3}^{i-1} j(i+2-j)\, a_j a_{i+2-j},$$

$$B_i = (ni/2)(7n - 4 - R)(i - \mu), \quad 3 \leqslant i \leqslant E.$$

If k takes on the values (1.3), then, the values of n and μ are equal, respectively, to

$$n = n_1 = (21 + 2\cdot 21^{1/2})/51, \quad \mu = 4,$$
$$n = n_2 = (56 + 25\cdot 2^{1/2})/23, \quad \mu = 5,$$
$$n = n_3 = (35 + 3\cdot 91^{1/2})4/29, \quad \mu = 6.$$

Then, the coefficients a_4, a_5, a_6 correspondingly revert to infinity; therefore, the following logarithmic terms must enter into the expansion (2.3):

$$q = \sum_{i=0}^{\mu-1} a_i \zeta_c^{3-i} \Delta^i + \sum_{i=0}^{1} (b_{\mu+i} \ln|\Delta| + a_{\mu+i}) \times$$

$$\times\ \zeta_c^{3-\mu-i} \Delta^{\mu+i} + O\left[\Delta^{\mu+2}(\ln|\Delta|)^2\right],$$

$$\mu = 4, 5, 6, \tag{2.5}$$

where $a_i (0 \leqslant i \leqslant \mu - 1)$ are determined by formulas (2.4); a_μ is an arbitrary constant; the remaining coefficients have the form

$$b_\mu = A_\mu/(7n^2 - 4n), \quad b_{\mu+1} = \{-b_\mu(n\mu - 3n + 2)^2 +$$
$$+ 3\mu(\mu+1) a_3 b_\mu\}/B_{\mu+1}, \quad a_{\mu+1} = \{A_{\mu+1} - b_\mu[4n +$$
$$+ 2n^2(\mu - 3) - 3(2\mu + 1)a_3] - n b_{\mu+1}(14n - 8 - R)\}/B_{\mu+1}.$$

With $\zeta > \zeta_c - 0$, the curve S is described by the expansion (2.5), where a_μ is a concrete number, previously unknown (μ = 4, 5, 6). To determine this number, Eq. (2.1) was integrated from the point $\zeta = -\infty$, with the initial data (2.2), to the point $\zeta = \zeta_c$. Calculations carried out in an M220M digital computer show that the curve S corresponds to the values

$$\mu = 4,\ a_\mu = 2.73; \quad \mu = 5,\ a_\mu = -0.0298; \tag{2.6}$$
$$\mu = 6,\ a_\mu = -0.00811.$$

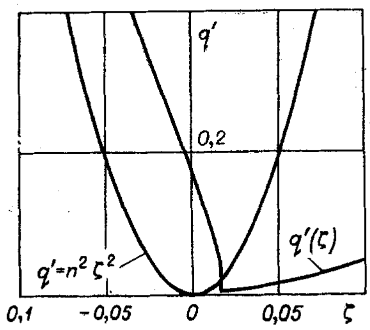

Fig. 1

3. With $\zeta > \zeta_c$, we denote by S the integral curve of Eq. (2.1), which, with $\zeta \to \zeta_c + 0$, is determined by the expansion (2.5) with the value of the coefficient given in (2.6). For prolongation of the flow beyond the characteristic C_0^-, we shall use the curve S.

Up to the limiting characteristic C_0^-, in all three cases (1.3) the flow is accelerated, passing through the speed of sound, and is then decelerated, remaining supersonic. Beyond the limiting characteristic, the behavior of the flows is different.

With k = 1.15465, the flow at first continues to be slowed, and then is accelerated; here, $A_2 = 0.532A_1$. The behavior of the function $q'(\zeta)$, characterizing the change in the longitudinal component along the straight line r = const, is close to that illustrated in Fig. 43 of [3]. In the calculations it was assumed that $A_1 = 2.15465$.

With k = 1.49647, a limiting line appears in the flow, which cannot be eliminated by a jump in the density. With k = 1.77208, a shock wave is generated at the center of the nozzle; it is then carried downstream. We note that $A_2 = 0.0867A_1$. A curve of the function q' is given in Fig. 1. In the calculations, it was assumed that $A_1 = 8.31623$. The equation of the shock wave is $\xi = 0.0177$.

If the flow is prolonged beyond the limiting characteristic C_0^-, using an arbitrary integral curve differing from S, then, in all three cases (1.3) there is the possibility of obtaining either continuous flows, or flows with shock waves, whose existence was demonstrated in [4].

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